Some remarks on strongly coupled systems of convection-diffusion equations in 2D

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Abstract

Almost nothing is known about the layer structure of solutions to strongly coupled systems of convection-diffusion equations in two dimensions. In some special cases we present first results.

AMS subject classification: 65 L10, 65 L12, 65 L50

1 Introduction

In the survey paper [4] the authors present some results for weakly coupled systems in one and two space dimensions but state concerning strongly coupled convection-diffusion problems ,,we have only a limited grasp of the situation". We aim to provide an inside into the nature of such problems at least in some special cases.

A practical example of strongly coupled systems of convection-diffusion equations in 2D (related to magnetohydrodynamic duct flow) is numerically studied in [2], namely

$$\begin{aligned} -\varepsilon\Delta u + a\nabla b &= f_1, \\ -\varepsilon\Delta b + a\nabla u &= f_2 \end{aligned} \tag{1.1}$$

with some boundary conditions.

Let us more generally consider the vector-valued function $u = (u_1, u_2)^T$ solving the system

$$-\varepsilon\Delta u + A_1\frac{\partial u}{\partial x_1} + A_2\frac{\partial u}{\partial x_2} + \rho\, u = f \qquad \text{in } \Omega, \tag{1.2a}$$

$$u = 0 \qquad \text{on } \Gamma = \partial\Omega, \tag{1.2b}$$

where ε is a small positive parameter. We assume the matrices A_1, A_2 to be symmetric and C^1 and that the unit outer normal $\nu = (\nu_1, \nu_2)$ to Ω exists a.e. on $\partial\Omega$.

Because

$$(\sum A_i\frac{\partial u}{\partial x_i}, u) = \frac{1}{2}\int_\Gamma (\nu\cdot Au, u)d\Gamma - \frac{1}{2}\big((\operatorname{div} A)u, u\big)$$

with

$$\operatorname{div} A = \sum \frac{\partial A_i}{\partial x_i},$$

it is standard to assume

$$\rho > \frac{1}{2} \sup_{\Omega} \| \operatorname{div} A \|_\infty. \tag{1.3}$$

Then, problem (1.2) admits a unique weak solution.

To describe the reduced problem we introduce the matrix

$$B := \nu_1 A_1 + \nu_2 A_2 \tag{1.4}$$

Suppose B to be nonsingular, i.e., $\partial\Omega$ to be noncharacteristic. Then B allows the decomposition

$$B = B^+ + B^-, \tag{1.5}$$

where B^+ is positive semidefinite, B^- negative semidefinite and the eigenvalues of B^+ are the positive eigenvalues of B and 0. The reduced problem to (1.2) is then given by

$$A_1 \frac{\partial u_0}{\partial x_1} + A_2 \frac{\partial u_0}{\partial x_2} + \rho\, u_0 = f \qquad \text{with } B^- u_0 = 0 \qquad \text{on } \Gamma. \tag{1.6}$$

In [12] it was proved that u converges for $\varepsilon \to 0$ to u_0. But concerning the convergence rate we only know the result of [5] for a problem with different boundary conditions: for $f \in H^1$ one has in the L_2 norm

$$\|u - u_0\|_0 \le C \varepsilon^{1/2} \|f\|_1. \tag{1.7}$$

In the literature not much is known about the structure of layers. In the next Section we will discuss this question in, at least, some special cases and consider problems with two small parameters, as well.

2 The reduced problem and location of layers

Let us first revisit the situation in 1D studied in [8]. We take the example

$$-\varepsilon u'' - \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} u' = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{2.1}$$

with $u(0) = u(1) = 0$. In the one parameter case the eigenvalues and eigenvectors of A determine the asymptotic structure of the solution. Here, the eigenvalues are 5 and -5 with the corresponding eigenvectors $(2,1)^T$ and $(1,-2)^T$. With that knowledge and the general solution of the reduced equation we made in [8] the Ansatz

$$u^0_{as} = w_0 + w_h \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + d_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} \exp(-5x/\varepsilon) + d_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \exp\big(-5(1-x)/\varepsilon\big). \tag{2.2}$$

The four constants c_1, c_2, d_1, d_2 are computed from the boundary conditions. Eliminating all exponentially small terms one gets indirectly the solution of the reduced problem, for our example

$$u_0 = \begin{pmatrix} 11/25\,x - 8/25 \\ -2/25\,x - 4/25 \end{pmatrix}.$$

Observe that u_0 does not satisfy either of the given boundary conditions.

There are two possibilities to derive the boundary conditions for the reduced equation *directly*. First as discussed in the Introduction, we can decompose the matrix B.

In our example we have

$$(Av, v) = 3v_1^2 + 8v_1v_2 - 3v_2^2 = (2v_1 + v_2)^2 - (v_1 - 2v_2)^2.$$

Therefore, the reduced problem is given by

$$-\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} u_0' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

with the boundary conditions

$$(2u_0^1 + u_0^2)(1) = 0 \quad \text{and} \quad (u_0^1 - 2u_0^2)(0) = 0. \tag{2.3}$$

An alternative approach consists in the diagonalization of the matrix A. Set

$$u = Tv \quad \text{or} \quad v = T^{-1}u.$$

After transformation $T^{-1}AT$ is a diagonal matrix, we obtain a decoupled system in v. In our example,

$$u = Tv \quad \text{with} \quad T = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

generates

$$-\varepsilon v_1'' - 5v_1' = \frac{4}{\sqrt{5}}, \tag{2.4a}$$

$$-\varepsilon v_2'' + 5v_2' = -\frac{3}{\sqrt{5}}. \tag{2.4b}$$

¿From the sign of the coefficients of the first order derivatives one can conclude where the layers of v are located. In our example v_1 has a layer at $x = 0$, v_2 at $x = 1$. The back transformation yields:

- the correct boundary conditions for the reduced problem are given by (2.3)
- both components u_1 and u_2 of the solution have layers at $x = 0$ and $x = 1$.

The transformation technique can also be applied for special systems in two dimensions. Let us assume that the constant matrices A_1, A_2 of the system (1.2) admit the representation

$$A_1 = T\ \mathrm{diag}(\lambda_1^1, \lambda_1^2))\, T^{-1}, \quad A_2 = T\ \mathrm{diag}(\lambda_2^1, \lambda_2^2))\, T^{-1} \tag{2.5}$$

with the orthogonal matrix

$$T = \begin{pmatrix} \sin\phi & \cos\phi \\ -\cos\phi & \sin\phi \end{pmatrix}$$

for some $\phi \in [0, 2\pi]$. Remark that this is the case if and only if $A_1A_2 = A_2A_1$.

Then, the transformation

$$u = Tv \quad \text{or} \quad v = T^{-1}u$$

yields also a decoupled system, namely (for $\rho = 0$, but this is not essential)

$$-\varepsilon\Delta v_1 + \lambda_1^1 \frac{\partial v_1}{\partial x_1} + \lambda_2^1 \frac{\partial v_1}{\partial x_2} = \tilde{f}_1 \tag{2.6a}$$

$$-\varepsilon\Delta v_2 + \lambda_1^2 \frac{\partial v_2}{\partial x_1} + \lambda_2^2 \frac{\partial v_2}{\partial x_2} = \tilde{f}_2. \tag{2.6b}$$

That means: the sign of $(\lambda_1^1, \lambda_2^1)^T \cdot \nu$ determines the location of the layers of v_1, the sign of $(\lambda_1^2, \lambda_2^2)^T \cdot \nu$ the location of the layers of v_2. Let us introduce

$$\Gamma_1^+ = \{x \in \Gamma : \ (\lambda_1^1, \lambda_2^1) \cdot \nu > 0\}$$

and analogously $\Gamma_1^-, \Gamma_1^0, \Gamma_2^+, \Gamma_2^-, \Gamma_2^0$. Then, the conditions

$$T_{r1}^{-1} u|_{\Gamma_1^-} = 0 \quad \text{and} \quad T_{r2}^{-1} u|_{\Gamma_2^-} = 0$$

(T_{r1}^{-1} denotes the first row of T^{-1}) are the correct boundary conditions for the reduced equation.

Assume $\Omega = (0,1)^2$. Then we have three typical cases (see Figure 1).

- (i) all λ are positive
  Then u_1 and u_2 have overlapping layers at $x_1 = 1$ and at $x_2 = 1$.
- (ii) only λ_1^2 is negative
  Then we have again overlapping layers at $x_2 = 1$, but ordinary layers of u_1 and u_2 at $x_1 = 0$ and at $x_1 = 1$.
- (iii) both components of λ_1 and λ_2 have a different sign
  Then u_1 and u_2 have ordinary layers at every edge of the unit square.

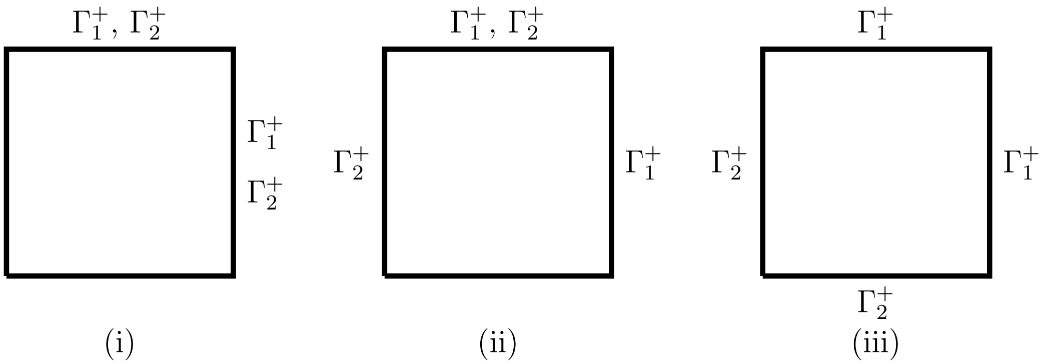

Figure 1: Location of boundary layers

Let us remark that the duct flow example mentioned in the introduction under the assumption $a < 0$ of [2] falls under case (iii) with

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Let us now consider the two parameter case

$$-\mathcal{E}\Delta u + A_1 \frac{\partial u}{\partial x_1} + A_2 \frac{\partial u}{\partial x_2} + \rho\, u = f \qquad \text{in } \Omega \tag{2.7a}$$

$$u = 0 \qquad \text{on } \Gamma = \partial\Omega \tag{2.7b}$$

with $\mathcal{E} = \operatorname{diag}(\varepsilon_1, \varepsilon_2)$ and two small positive parameters. We are mostly interested in the case $\varepsilon_1/\varepsilon_2 \ll 1$.

Let us first collect some facts from [9] for the one-dimensional system

$$-\operatorname{diag}(\varepsilon_1, \varepsilon_2)u'' - Bu' + Au = f, \quad u(0) = u(1) = 0. \tag{2.8}$$

Assume

$$B = \begin{pmatrix} -b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \quad \text{with} \quad b_{11} > 0 \quad \text{and } \hat{D} = b_{11}b_{22} + b_{12}^2 > 0.$$

Surprisingly, not the eigenvalues of B generate the structure of the boundary layers. The layer at $x = 0$ is characterized by the exponential $\exp(-b_{11}/\varepsilon_1)$, but the layer at $x = 1$ by the exponential $\exp(\frac{\hat{D}}{b_{11}}\frac{1}{\varepsilon_2})$.

The quantities $-b_{11}$ and $\frac{-\hat{D}}{-b_{11}}$ are quotients of the leading principal minors of the matrix B. These values arise as well in the LDL^T decomposition of the matrix B. In accordance with that observation the authors of [6] proposed to introduce new variables v by $v = L^T u$ resulting in

$$-\mathcal{E}^* v'' - \operatorname{diag}(d_1, d_2)v' + A^* v = f^*, \quad v(0) = v(1) = 0 \tag{2.9}$$

with

$$\mathcal{E}^* = -L^{-1}\mathcal{E}(L^{-1})^T.$$

Unfortunately, the matrix $\mathcal{E}^*$ is not diagonal. But its structure

$$\mathcal{E}^* = \begin{pmatrix} \varepsilon_1 & -\varepsilon_1\, l \\ -\varepsilon_1\, l & \varepsilon_1\, l^2 + \varepsilon_2 \end{pmatrix}$$

tells us (remember $\varepsilon_1/\varepsilon_2 \ll 1$) the following: Assuming, for instance, $d_1 < 0$ and $d_2 > 0$, both solution components have strong layers at $x = 0$ of the structure $\exp\big((d_1/\varepsilon_1)x\big)$. However, at $x = 1$ a *weak* layer of $v_2 = u_2$ of the structure $O(\varepsilon_1/\varepsilon_2)\exp\big(-(d_2/\varepsilon_2)(1-x)\big)$ generates a strong layer of v_1 and, consequently, u_1. In [9] this solution behavior was derived from some asymptotic approximation.

The signs of d_1, d_2 also define the boundary conditions of the reduced problem, under our assumptions

$$u_0^2(1) = 0, \quad \text{and} \quad u_0^1(0) + l u_0^2(0) = 0. \tag{2.10}$$

Consider the example

$$-\operatorname{diag}(\varepsilon_1, \varepsilon_2)u'' - \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} u' = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

with $u(0) = u(1) = 0$. Here

$$L = \begin{pmatrix} 1 & 0 \\ 4/3 & 1 \end{pmatrix}$$

and $D = \operatorname{diag}(3, -25/3)$. Consequently, in the case $\varepsilon_1 \ll \varepsilon_2$ we conclude

- the boundary conditions for the reduced problem are

$$u_0^2(1) = 0, \quad \text{and} \quad u_0^1(0) + 4/3 u_0^2(0) = 0$$

- the solution u is characterized by the layer terms

$$\begin{pmatrix} 4 \\ -3 \end{pmatrix} \exp(-\frac{3}{\varepsilon_1}x) \quad \text{and} \quad \begin{pmatrix} 3 \\ O(\varepsilon_1/\varepsilon_2) \end{pmatrix} \exp\Big(-\frac{25}{3\varepsilon_2}(1-x)\Big).$$

Remark 1 *In the one parameter case the decomposition* (1.5) *allows to define the boundary conditions for the reduced problem. But with two parameters, for instance in 1D, the matrix* $\mathcal{E}^{-1}B$ *is not symmetric. Symmetrization of that matrix with a transformation of the type*

$$T^{-1}(\mathcal{E}^{-1}B)T \quad \textit{with} \quad T = \begin{pmatrix} 1 & \mu \\ (\frac{\varepsilon_1}{\varepsilon_2})^{1/2}\mu & 1 \end{pmatrix}$$

and adequately chosen μ *is possible. But it is simpler to apply the* LDL^T *decomposition described above.* $\square$

The results for the two parameter problem (2.8) can now be generalized to (2.7) if we assume

$$A_1 = L\ \mathrm{diag}(d_1^1, d_1^2)\, L^T, \quad A_2 = L\ \mathrm{diag}(d_2^1, d_2^2)\, L^T. \tag{2.11}$$

The transformation

$$u = (L^T)^{-1} v \quad \text{or} \quad v = L^T u$$

yields the system

$$\begin{aligned} -\varepsilon_1 \Delta v_1 - \varepsilon_1 q \Delta v_2 + d^1 \cdot \nabla v_1 + \tilde{\rho}(v_1 + \alpha v_2) &= \tilde{f}_1, \\ -\varepsilon_1 q \Delta v_1 - (\varepsilon_1 q^2 + \varepsilon_2) \Delta v_2 + d^2 \cdot \nabla v_2 + \rho v_2 &= \tilde{f}_2. \end{aligned} \tag{2.12}$$

Next we introduce as above $\Gamma_1^-, \Gamma_1^+, \Gamma_2^-, \Gamma_2^+$ with respect to the vectors d^1, d^2. Both components v_1 and v_2 as well as u_1 and u_2 have strong exponential layers related to ε_1 on Γ_1^+. But on Γ_2^+ the component $v_2 = u_2$ has only a weak layer. That weak layer generates a strong layer of u_1 related to the larger parameter ε_2.

The correct boundary conditions for the reduced equation are

$$L_{r1}^T u|_{\Gamma_1^-} = 0 \quad \text{and} \quad u_2|_{\Gamma_2^-} = 0.$$

Summarizing: Under some restrictive conditions it is possible to characterize the possible layers of strongly coupled convection-diffusion systems in 2D. Moreover, the adequate boundary conditions for the reduced problem can be derived (important for numerical methods which first solve the simpler reduced problem).

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